This solution can be represented by a series in powers of $\varepsilon^{1/3}$ converging uniformly and absolutely for small $|\varepsilon| < \varepsilon_0$.

The proof of these theorems is not included here. It is only noted that the procedure is similar to that used in [6, 7].

The above theorems imply that the series for $\Phi(\theta, \varepsilon)$ and $\tau(\theta, \varepsilon)$ converge uniformly and absolutely. Convergence of the series in powers of ε and $\varepsilon^{1/3}$ (in the case of $\nu = \nu_1$) for integrand functions in (1.4) follows from the general theorems of analysis on substitution of one series into the other. The general theorems of analysis also yield the proof of convergence of the series (4.1), (4.2) and (4.3).

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METHODS OF MECHANICS OF A CONTINUOUS MEDIUM FOR THE DESCRIPTION OF MULTIPHASE MIXTURES

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The principal assumptions in the construction of a general multivelocity model of a continuous multiphase medium are examined and the fundamental equations (for mass, momentum and energy) of mechanics are obtained for each phase in the heterogeneous mixture. On the basis of these equations a closed system is proposed which describes the motion of a dispersed mixture of two compressible phases in the presence of phase changes. Energy transitions in phase transformations are analyzed. The fundamental relationships on the surface of the discontinuity are derived. Proceeding from the assumption of additivity of the internal energy of the mixture according to masses of components and from the assumption of local equilibrium within the limits of a phase, thermodynamic questions of a heterogeneous mixture are analyzed. In particular, an explicit expression is obtained for the dissipation function.

The basic ideas of utilizing interpenetrating multivelocity continua in the mechanics of mixtures were worked out in [1] and [2]. Questions of derivation of corresponding equations by means of averaging methods were examined in [3], where the earlier literature is also mentioned. It should be noted that some equations of multivelocity continua were also contained in [4, 5].

It is necessary [6] to distinguish homogeneous or multicomponent (*) mixtures (solutions, alloys, mixture of gases) from heterogeneous or multiphase mixtures (emulsions, suspensions, gaseous suspensions, soil saturated with water, mixtures of powders, etc.). In the homogeneous mixture each constituent (component) can be considered as occupying the entire volume of the mixture

$$V_1 = V_2 = \ldots = V_m = V$$

while each constituent (phase) in the heterogeneous mixture occupies only part of the volume of the mixture, so that

$$V_1 + V_2 + \ldots + V_m = V$$

A neglect of this situation and also the introduction of a temperature concept of the mixture in the case where the temperatures of the constituents are not equal, lead the authors of [7–9] to formal theories which are not supported by physical reality, at least in the case of heterogeneous media [10].

The results obtained in this work are generalizations of [11] and [12], where an analogous mixture was examined, however, for the case where one of the phases is incompressible (in [11] in the absence of phase transitions). As an example for the case where it is necessary to take into account the compressibility of both phases, we point out the flow of dispersed vapor-liquid mixtures at pressures and temperatures close to the critical values. Another area of application of results presented, is the investigation of propagation of strong shock waves in condensed heterogeneous mixtures.

We note the work [13] where a model of two compressible phases with application to water saturated soil but in the absence of phase transitions was examined. We also note the work [14] where a single-velocity formulation was used and where equations of motion were obtained which are applicable to a two-phase solid body taking into consideration phase transitions.

1. Fundamental assumptions. The motion of the multiphase mixture will be examined under the following fundamental assumption. The distances over which the flow parameters change substantially (outside the surface of discontinuity) are much greater than the characteristic dimensions of inhomogeneities or inclusions. This permits to describe the multiphase mixture, just as the homogeneous mixture, in the form of a sum total m (with respect to the number of phases) of continua which occupy one and the same volume. Then at each point of the volume which is occupied by the mixture we can introduce average densities $\rho_1, \ldots, \rho_m$ characterizing the mass of the phase

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*) Unfortunately, the term "multicomponent mixture" is sometimes used for heterogeneous (multiphase) mixtures.

in the unit volume of the mixture. We can introduce velocities $\mathbf{v}_1, \ldots, \mathbf{v}_m$ etc., and also write the conservation equations for mass, momentum, and energy for each component. These equations will contain terms which characterize the interaction (due to mass, force, and energy) between the components. Further the theories of homogeneous and heterogeneous mixtures take separate approaches. The latter utilizes quantities which characterize the parts of the volume of the mixture α_i $(i = 1, \ldots, m)$ which are occupied by each phase

$$\alpha_1 + \ldots + \alpha_m = 1 \qquad (\alpha_i \geqslant 0) \tag{1.1}$$

In this manner, in addition to the average densities ρ_i, the true densities of matter in phases ρ_i° (mass of ith phase per unit volume of ith phase) are determined

$$\rho_i^\circ = \rho_i / \alpha_i \tag{1.2}$$

The mechanics of heterogeneous media must take into account the fact that the components in the mixture are present in the form of macroscopic (with respect to molecular dimensions) inclusions or a medium which surrounds these inclusions so that many mechanical and thermodynamic properties of the ith phase $(i = 1, \ldots, m)$ do not depend directly on the presence of other phases. However, the deformation of each phase which determines the state and the reaction forces of the phase is determined not only by the displacement of the external boundaries (the velocity field $\mathbf{v}_i$) of the phase, but also by the displacement of interphase surfaces within the selected volume.

In the general case it is necessary for each component to examine not only the external tensor of deformation velocities

$$e_i^{kl} = \frac{1}{2}\left(\frac{\partial v_i^k}{\partial x^l} + \frac{\partial v_i^l}{\partial x^k}\right) \tag{1.3}$$

but also an entire set ($m - 1$ for each phase) of certain tensors $(\delta_{ij}^{kl},\ j = 1, \ldots, m,$ $j \neq i)$ which take into account the displacement of matter of the ith phase on the surfaces of phase separation so that the true velocity of phase deformation is determined by the tensor

$$e_i^{\circ kl} = e_i^{kl} + \sum_{j=1, j\neq i}^{m} \delta_{ij}^{kl} \tag{1.4}$$

The determination of tensors δ_{ij} is each time connected with the application of conditions of simultaneous motion and deformation of phases, and with conditions which take into account the structure of components (the shape and the size of inclusions, their distribution, etc.). In those cases where the effects of strength do not have any significance (gaseous suspensions, liquid with bubbles or particles, solid bodies under conditions of very high pressures), the conditions of simultaneous motion are substantially simpler than in the general case. In essence, they reduce to giving equations which determine the volume content of phases α_i. We note that the most commonly encountered equation of this type is the equation of pressure equality of phases.

2. Integral equations. Let us examine a volume V which is fixed in some inertial system of coordinates and bounded by the surface S. The equations for conservation of mass for the first and second phases inside the volume V have the form

$$\int_V \frac{\partial \rho_1}{\partial t} dV = -\int_S \rho_1 v_1^n dS + \int_V (J_{21} - J_{12})\, dV, \quad \rho_1 = \rho_1^\circ \alpha_1 \tag{2.1}$$

$$\int_V \frac{\partial \rho_2}{\partial t} dV = -\int_S \rho_2 v_2^n dS + \int_V (J_{12} - J_{21}) dV, \quad \rho_2 = \rho_2^\circ \alpha_2 \quad \alpha_1 + \alpha_2 = 1$$

Here J_{12} and J_{21} are "observable" macroscopic velocities of phase transformation each of which is nonnegative ao that J_{ij} gives only the rate of formation (in the opposite case $J_{ij} = 0$) of the jth phase at the expense of the ith phase per unit volume and time. This means that, by convention, phase transitions are broken up into two reactions (each of which from the kinetic view point has a forward and reverse reaction). This is related to the fact that the two possible resulting directions of phase transformations $1 \to 2$ and $2 \to 1$ lead to different changes in momentum and energy of the individual phase. These changes cannot be taken into account through a change in sign of the reaction rate (which is sufficient when the equations of momentum and energy are written for the entire mixture (see below)). Combining the integral equations (2.1), we have the equation of conservation of mass of the entire mixture

$$\int_V \left(\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_2}{\partial t}\right) dV = -\int_S (\rho_1 v_1{}^n + \rho_2 v_2{}^n)\, dS \tag{2.2}$$

The equations of conservation of momentum of each phase separately have the following form:

$$\begin{gathered}\int_V \frac{\partial}{\partial t}(\rho_1 \mathbf{v}_1)\, dV = -\int_S \rho_1 \mathbf{v}_1 v_1{}^n dS + \int_S \sigma_1{}^n dS + \\ + \int_V \mathbf{r}_{21} dV + \int_V (J_{21}\mathbf{v}_{21} - J_{12}\mathbf{v}_{12})\, dV + \int_V \rho_1 \mathbf{F}_1 dV \\ \int_V \frac{\partial}{\partial t}(\rho_2 \mathbf{v}_2)\, dV = -\int_S \rho_2 \mathbf{v}_2 v_2{}^n dS + \int_S \sigma_2{}^n dS - \int_V \mathbf{r}_{21} dV + \\ + \int_V (J_{12}\mathbf{v}_{12} - J_{21}\mathbf{v}_{21})\, dV + \int_V \rho_2 \mathbf{F}_2 dV\end{gathered} \tag{2.3}$$

Here $\mathbf{r}_{21}$ is the volume force related to the unit volume of the mixture and resulting from the interaction between phases within the volume V due to forces of friction, pressure, coupling between phases, the effect of associated mass, etc. The fourth terms in the right sides of (2.3) represent the change of momentum of the corresponding phase as a result of phase transformations. For example, the transition $2 \to 1$ results in the momentum $J_{21}\mathbf{v}_{21}$ transferring from the second phase into the first phase. The transition $1 \to 2$ corresponds to the transfer of momentum $J_{12}\mathbf{v}_{12}$ from the first phase to the second phase. This means that $\mathbf{v}_{12}$ and $\mathbf{v}_{21}$ are velocities of masses which suffer phase transitions corresponding to $1 \to 2$ and $2 \to 1$. Furthermore, $\mathbf{F}_1$ and $\mathbf{F}_2$ are external mass forces which act on the first and second phases. The second terms in the right sides of (2.3) are connected with the interaction of the medium, which is external with respect to the surface S, with the corresponding phase. This interaction is characterized by the tensors of surface forces $\sigma_i{}^{kl}$ $(i = 1, 2)$ which can be represented as

$$\sigma_i{}^{kl} = \alpha_i \sigma_i{}^{\circ kl} + \alpha_j \sigma_{ji}{}^{*kl} \tag{2.4}$$

here $\sigma_i{}^{\circ kl}$ is the actual stress tensor connected with the action of surface forces on the ith phase from the same ith phase on the external boundaries of the selected volume of the mixture. The tensor $\sigma_{ji}{}^{*kl}$ is connected with interphase surface forces acting on the ith phase from the jth phase on the same external boundaries.

Combining Eqs. (2.3), we have the equation of conservation of momentum for the entire mixture

$$\int\limits_V \frac{\partial}{\partial t}(\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2)\, dV = -\int\limits_S (\rho_1 \mathbf{v}_1 v_1^n + \rho_2 \mathbf{v}_2 v_2^n)\, dS + \\ + \int\limits_S (\sigma_1^n + \sigma_2^n)\, dS + \int\limits_V (\rho_1 \mathbf{F}_1 + \rho_2 \mathbf{F}_2)\, dV \tag{2.5}$$

The energy of the mixture is composed of the internal and kinetic energies of the components. In this connection we will consider each phase as locally homogeneous. That is, in each elementary volume of the mixture the substance of each phase which also contains the inclusions (drops, particles, bubbles, etc.) is taken as homogeneous throughout to the very interface of phases. For this reason the energy of each component is considered proportional to its mass. This is equivalent to the statement that singularities of the surface layer of material which forms the boundaries of phase separation and which has a thickness of the order of the radius of molecular interactions ($\sim 10^{-9}$ m), are not taken into account further.

For this it is necessary that the dimensions of the inclusions be many times greater than the thickness of this layer. Furthermore, only that part of kinetic energy will be taken into account which is connected with the macroscopic motion of phases with velocities $\mathbf{v}_i$. In fact there are also small-scale flows (with characteristic linear dimensions which have the same order of magnitude as the inhomogeneities of the mixture. Examples of such flows are pulsating motions around bubbles, reverse flows carrying the fluids near the inclusions). In the existing theories of interpenetrating motion the kinetic energy of such flows is not taken into account. In this manner the case is examined where with homogeneous representation of the energy of phases, the energy of the mixture is additive with respect to the mass of phases. A separate paper by the author will be devoted to the problem of consideration of surface effects within the framework of concepts of Gibbs in the fluid mechanics of a mixture. In this paper the kinetic energy of the small-scale motion of phases will also be considered.

Let us now write the energy equations separately for the first and second phases within the volume V

$$\int\limits_V \frac{\partial}{\partial t}\left[\rho_1\left(u_1 + \frac{v_1^2}{2}\right)\right] dV = -\int\limits_S \rho_1\left(u_1 + \frac{v_1^2}{2}\right) v_1^n dS + \int\limits_S C_1^n dS - \int\limits_S q_1^n dS + \\ + \int\limits_V \rho_1 (\mathbf{F}_1 \mathbf{v}_1 + Q_1)\, dV + \int\limits_V \left[q_{21} + w_{21} + J_{21}\left(u_{21} + \frac{v_{21}^2}{2}\right) - J_{12}\left(u_{12} + \frac{v_{12}^2}{2}\right)\right] dV \\ \int\limits_V \frac{\partial}{\partial t}\left[\rho_2\left(u_2 + \frac{v_2^2}{2}\right)\right] dV = -\int\limits_S \rho_2\left(u_2 + \frac{v_2^2}{2}\right) v_2^n dS + \int\limits_S C_2^n dS - \int\limits_S q_2^n dS + \\ + \int\limits_V \rho_2 (\mathbf{F}_2 \mathbf{v}_2 + Q_2)\, dV - \int\limits_V \left[q_{21} + w_{21} + J_{21}\left(u_{21} + \frac{v_{21}^2}{2}\right) - J_{12}\left(u_{12} + \frac{v_{12}^2}{2}\right)\right] dV \tag{2.6}$$

Here u_i is the internal energy of the ith phase. The second and third terms in the right sides of (2.6) correspond to the work of external surface forces acting on the ith phase on the surface S, characterized by vectors $\mathbf{C}_i$, and to the influx of heat from outside to the ith phase through the surface S. The influx of heat is characterized by vectors $\mathbf{q}_i$. The fourth terms represent the work of external mass forces and also the strength of external heat sources located inside the volume V. The last terms in the right side describe the energy exchange related to the unit volume of mixture and unit time within the volume V between the first and second phases due to heat exchange

(q_{21}), the work of interaction forces (forces of friction, pressure, cohesion, etc.) between phases (w_{21}), and phase transitions. The transition $2 \rightarrow 1$ leads to the situation where the energy of the first phase increases by the quantity $J_{21}(u_{21} + {}^1/_2\, v_{21}^2)$ at the expense of the second phase. The transition $1 \rightarrow 2$ leads to the situation where the energy of the second phase increases by the quantity $J_{12}(u_{12} + {}^1/_2 v_{12}^2)$ at the expense of the first phase. That is, u_{12} and u_{21} are internal energies of masses which transfer from state 1 into state 2 and from state 2 to state 1.

Combining Eqs. (2.6), we obtain the energy conservation equation of the mixture inside the volume V. This energy can change only as a result of the action of external sources

$$\int_V \frac{\partial}{\partial t}\left[\rho_1\left(u_1 + \frac{v_1^2}{2}\right) + \rho_2\left(u_2 + \frac{v_2}{2}\right)\right] dV =$$

$$= -\int_S \left[\rho_1\left(u_1 + \frac{v_1^2}{2}\right) v_1^n + \rho_2\left(u_2 + \frac{v_2^2}{2}\right) v_2^n\right] dS + \int_S (C_1^n + C_2^n)\, dS -$$

$$- \int_S (q_1^n + q_2^n)\, dS + \int_V [\rho_1(\mathbf{F}_1\mathbf{v}_1 + Q_1) + \rho_2(\mathbf{F}_2\mathbf{v}_2 + Q_2)]\, dV \qquad (2.7)$$

3. Differential equations. Utilizing the formula of Gauss-Ostrogradskii in the integral equations and introducing the operator of substantive derivative for each phase (here and subsequently summation is performed only over the superscripts which are related to projections on coordinate axes)

$$\frac{d_i}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_i \boldsymbol{\nabla} \equiv \frac{\partial}{\partial t} + v_i^k \frac{\partial}{\partial x^k} \equiv \frac{\partial}{\partial t} + v_i^k \nabla^k \qquad (3.1)$$

we obtain the differential equations of conservation of mass, momentum and energy of each phase for continuous motion

$$\frac{\partial \rho_i}{\partial t} + \boldsymbol{\nabla}(\rho_i \mathbf{v}_i) = J_{ji} - J_{ij} \qquad (3.2)$$

$$\rho_i \frac{d_i \mathbf{v}_i}{dt} = \nabla^k \boldsymbol{\sigma}_i^k + \mathbf{r}_{ji} + J_{ji}(\mathbf{v}_{ji} - \mathbf{v}_i) - J_{ij}(\mathbf{v}_{ij} - \mathbf{v}_i) + \rho_i \mathbf{F}_i$$

$$\rho_i \frac{d_i}{dt}\left(u_i + \frac{v_i^2}{2}\right) = \boldsymbol{\nabla}\mathbf{C}_i + q_{ji} + w_{ji} + J_{ji}\left(u_{ji} - u_i + \frac{v_{ji}^2 - v_i^2}{2}\right) -$$

$$- J_{ij}\left(u_{ij} - u_i + \frac{v_{ij}^2 - v_i^2}{2}\right) - \boldsymbol{\nabla}\mathbf{q}_i + \rho_i(\mathbf{F}_i \mathbf{v}_i + Q_i)$$

$$i, j = 1, 2; \; i \neq j$$

We note that equations of conservation of moments of momentum in the case of absence of internal moments in each phase and in the absence of distributed couples in the mass and on the surface lead to symmetry of tensors σ_i^{kl} in analogy to the classical case.

From (2.2), (2.5) and (2.7) in analogy to (3.2), or combining the corresponding equations of (3.2), we can obtain equations for conservation of mass, momentum and energy of the mixture

$$\frac{\partial}{\partial t}(\rho_1 + \rho_2) + \boldsymbol{\nabla}(\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2) = 0 \qquad (3.3)$$

$$\rho_1 \frac{d_1 \mathbf{v}_1}{dt} + \rho_2 \frac{d\mathbf{v}_2}{dt} = \nabla^k(\boldsymbol{\sigma}_1^k + \boldsymbol{\sigma}_2^k) + (J_{12} - J_{21})(\mathbf{v}_1 - \mathbf{v}_2) + \rho_1 \mathbf{F}_1 + \rho_2 \mathbf{F}_2$$

$$\rho_1 \frac{d_1}{dt}\left(u_1 + \frac{v_1^2}{2}\right) + \rho_2 \frac{d_2}{dt}\left(u_2 + \frac{v_2^2}{2}\right) = \boldsymbol{\nabla}(\mathbf{C}_1 + \mathbf{C}_2) + (J_{12} - J_{21}) \times$$

$$\times\left(u_1-u_2+\frac{v_1^2-v_2^2}{2}\right)-\nabla(\mathbf{q}_1+\mathbf{q}_2)+\rho_1(\mathbf{F}_1\mathbf{v}_1+Q_1)+\rho_2(\mathbf{F}_2\mathbf{v}_2+Q_2)$$

From a comparison of (3.2) and (3.3) it is evident that in the directional relationship (2 → 1 or 1 → 2) of the resulting phase transformation in a given point, the force and energy interaction on an individual phase cannot be taken into account by a change in the sign of the reaction rate, because in the general case $\mathbf{v}_{12}\neq\mathbf{v}_{21}$, $\mathbf{u}_{12}\neq\mathbf{u}_{21}$ and the effects in the right sides of Eqs. (3.2), which depend on J_{12} and J_{21} are different. At the same time for the entire mixture the force and energy effects can be taken into account through a change in sign in the rate of transition because this effect enters into Eq. (3.3) through the difference $J_{12}-J_{21}$.

From the second relationship (3.2) we find the equation of kinetic energy for an individual phase

$$\rho_i\frac{d_i}{dt}\left(\frac{v_i^2}{2}\right)=\mathbf{v}_i\nabla^k\boldsymbol{\sigma}_i^k+\mathbf{r}_{ji}\mathbf{v}_i+J_{ji}(\mathbf{v}_{ji}\mathbf{v}_i-v_i^2)-J_{ij}(\mathbf{v}_{ij}\mathbf{v}_i-v_i^2)+\rho_i\mathbf{F}_i\mathbf{v}_i \tag{3.4}$$

the equation for the kinetic energy of the mixture

$$\begin{aligned}\rho_1\frac{d_1}{dt}\left(\frac{v_1^2}{2}\right)+\rho_2\frac{d_2}{dt}\left(\frac{v_2^2}{2}\right)&=v_1\nabla^k\boldsymbol{\sigma}_1^k+v_2\nabla^k\boldsymbol{\sigma}_2^k+\mathbf{r}_{12}(\mathbf{v}_2-\mathbf{v}_1)+\\&+(J_{12}-J_{21})\frac{v_1^2-v_2^2}{2}+J_{21}(\mathbf{v}_1-\mathbf{v}_2)(\mathbf{v}_{21}-{}^1/_2(\mathbf{v}_1+\mathbf{v}_2))-\\&-J_{12}(\mathbf{v}_1-\mathbf{v}_2)(\mathbf{v}_{12}-{}^1/_2(\mathbf{v}_1+\mathbf{v}_2))+\rho_1\mathbf{F}_1\mathbf{v}_1+\rho_2\mathbf{F}_2\mathbf{v}_2\end{aligned} \tag{3.5}$$

and the equation for the internal energy of the individual phase

$$\begin{aligned}\rho_i\frac{d_iu_i}{dt}&=\nabla\mathbf{C}_i-\mathbf{v}_i\nabla^k\boldsymbol{\sigma}_i^k+w_{ji}-\mathbf{r}_{ji}\mathbf{v}_i+q_{ji}+J_{ji}(u_{ji}-u_i)-J_{ij}(u_{ij}-u_i)+\\&+{}^1/_2J_{ji}(\mathbf{v}_{ji}-\mathbf{v}_i)^2-{}^1/_2J_{ij}(\mathbf{v}_{ij}-\mathbf{v}_i)^2-\nabla\mathbf{q}_i+\rho_iQ_i\end{aligned} \tag{3.6}$$

This equation is obtained from formally balanced relationships and its direct application to a concrete case is connected with considerable difficulties. In the absence of phase transitions the application to a concrete case was carried out in [13] with the help of some additional considerations. It will be shown below that in the general case it is better and clearer to proceed from the analogous relationship in the form of the equation of heat influx of the ith phase

$$\rho_i\frac{d_iu_i}{dt}=\rho_iA_i-J_{ji}x_{i\,ji}-J_{ij}x_{i,ij}+q_{ji}-\nabla\mathbf{q}_i+\rho_iQ_i \tag{3.7}$$

Here A_i represents the work of internal forces [15] in the unit mass of the ith phase per unit of time. The remaining terms represent the influx of heat. Here $x_{i,ji}$ and $x_{i,ij}$ are the heat fluxes from the ith phase to the material which undergoes the transformation $j\to i$ and $i\to j$, respectively. The heat fluxes are referred to the masses which undergo these transitions.

4. Evaluation of mechanical and thermodynamic properties of phases. Concrete application of the model to a two-phase continuous medium naturally requires the evaluation of mechanical and thermodynamic properties of the phases. Taking into account the material presented in Sect. 1, it will be assumed that the properties of each phase in the mixture are determined by the same relationships which apply assuming that this phase occupies the entire volume under the same conditions. In this connection the deformation enters into these relationships through the actual tensor of deformation or through the actual velocities of deformation. That is, knowing the properties of each phase, we have the equation for the actual stress tensor

$$\sigma_i^{*kl} = \sigma_i^{\circ kl}(\varepsilon_i^{\circ mn}, e_i^{\circ mn}, T_i, \chi_i^1, \ldots, \chi_i^p) \tag{4.1}$$

This tensor enters into the expression of the general tensor of surface forces of the ith phase (2.4). Here $\chi_i^1, \ldots, \chi_i^p$ are physical-chemical parameters of the ith phase.

Further we adopt the hypothesis of local equilibrium within the boundaries of the phase. This makes it possible for each phase to introduce its own temperature ($T_1, \ldots, T_m$) and other thermodynamic functions (entropy, enthalpy, internal energy and others) corresponding to the substance of the phase at its temperature T_i, deformation $\varepsilon_i^{\circ kl}$ and other physical-chemical parameters. For example

$$u_i = u_i(\varepsilon_i^{\circ kl}, T_i, \chi_i^1, \ldots, \chi_i^p) \tag{4.2}$$

The second law of thermodynamics for the ith phase in this case has the form

$$T_i d_i s_i = d_i q_i^{(e)} + d_i q_i', \qquad d_i q_i' \geqslant 0 \tag{4.3}$$

where s_i is the specific entropy of the ith phase, $d_i q_i^{(e)}$ is the external influx of heat and $d_i q_i'$ is the so-called uncompensated heat, all taken per unit mass of the ith phase. In this case $d_i q_i'$ is determined by the properties of the ith phase only.

In this manner the problem of multiphase motion in the case where the physical and mechanical properties of each phase are known, is reduced to the development of conditions for simultaneous motion and deformation, i.e. to the determination of δ_{ij}^{kl} and other terms characterizing the interaction of phases

$$(\sigma_{ij}^{*kl}, J_{ji}, J_{ij}, \mathbf{r}_{ij} = -\mathbf{r}_{ji}, \mathbf{v}_{ij}, \mathbf{v}_{ji}, u_{ij}, u_{ji}, q_{ij} = -q_{ji}, w_{ij} = -w_{ji})$$

5. Dispersed mixtures of two compressible phases. Let us examine a heterogeneous medium consisting of a mixture of two compressible fluids in each of which effects of strength are not present. The second phase is present in the form of separate inclusions of uniform size (particles, drops, bubbles). Direct interactions (for example collisions) between these can be neglected. The first phase is the supporting medium which is described through the model of a viscous fluid. In this case it can usually be assumed that

$$\sigma_1^{kl} = -p_1\delta^{kl} + \tau_1^{kl}, \qquad \sigma_2^{kl} = 0 \tag{5.1}$$

where δ^{kl} is the Kronecker symbol, p_1 is the pressure and τ_1^{kl} is the viscosity tensor of the first phase. For this tensor the Navier-Stokes relationship determined by the external tensor of deformation velocities (1.3) is assumed

$$\tau_1^{kl} = \lambda_1^* \nabla \mathbf{v}_1 + 2\mu_1^* e_1^{kl} \tag{5.2}$$

The effect of displacements on interphase boundaries, on which the actual velocity of deformation of the first phase (1.4) also depends, is taken into account through corrections (of the Einstein type) in the coefficients of viscosity λ_1^* and μ_1^*. These corrections take into account not only the properties of the supporting medium, but also the properties of inclusions [16]. In concentrated suspensions it is sometimes necessary to examine also the nonzero (because of random motion of particles) values [17, 18] of the stress tensor σ_2^{kl}.

In analogy to (5.1) and (5.2) it is possible in the absence of radiant heat transfer to take the heat transfer relationship within the supporting phase in the form

$$\underline{q_1} = -k_1^* \nabla T_1, \qquad q_2 = 0 \tag{5.3}$$

Taking into account the hypothesis of local equilibrium within the boundaries of the phase (Sect. 4) and assuming that the phases represent two-parameter media [15], i.e. the thermodynamic functions of each medium depend only on two thermodynamic parameters of state (for example, on the actual density ρ_i° and temperature T_i or the pressure p_i and the temperature T_i), we have

$$u_i = u_i(\rho_i^\circ, T_i), \quad p_i = p_i(\rho_i^\circ, T_i), \quad s_i = s_i(\rho_i^\circ, T_i) \tag{5.4}$$

In this case the Gibbs relationship is valid

$$T_i \frac{d_i s_i}{dt} = \frac{d_i u_i}{dt} + p_i \frac{d_i}{dt} \frac{1}{\rho_i^\circ} \qquad (i = 1,2) \tag{5.5}$$

For many problems where the characteristic time of the macroscopic process under examination is much greater than the characteristic time of establishment of interphase pressure equilibrium, we can (when the effects of surface tension are small (Sect. 2)) take advantage of the "single pressure" model, i.e.

$$p_1(\rho_1^\circ, T_1) = p_2(\rho_2^\circ, T_2) = p \tag{5.6}$$

In this case this represents the condition for simultaneous deformation of phases. This condition controls the volume sizes of phases. In many cases we can use as such a condition the incompressibility of one of the phases. In the case of nonequilibrium with respect to pressure between the phases, it is necessary to specify the kinetic equation which relates p_1 and p_2 (of the type of the equation of Lamb for a bubble) and to take into account in this connection the kinetic energy of pulsating motion of phases. This energy is converted into compression energy and back. The process can be accompanied by dissipation because of viscosity.

The force of interaction between the supporting medium and the inclusions is represented in the form

$$\mathbf{r}_{12} = -\alpha_2 \nabla p_1 + \mathbf{f}_{12} \tag{5.7}$$

Here the first term is related to interactions of the pressure field with the inclusions (Archimedes force). The second term is related to velocity nonequilibrium between phases (difference between $\mathbf{v}_1$ and $\mathbf{v}_2$) which in turn in this case is due to three effects

$$\mathbf{f}_{12} = \mathbf{f}_f + \mathbf{f}_m + \mathbf{f}_r \tag{5.8}$$

Here $\mathbf{f}_f$, the friction force (Stokes force), which is due to viscous forces in the interaction between phases, is determined by the difference in velocities (slip) $\mathbf{v}_1 - \mathbf{v}_2$, the size, the quantity and the shape of inclusions, and also the physical properties of phases. Further $\mathbf{f}_m$ is the force connected with the influence "of associated masses" and arising because of accelerated motion of the inclusion with respect to the supporting phase when disturbances arise in the latter over distances of the order of dimensions of the inclusions. It is these small-scale disturbances which result in an additional pressure force which is not taken into account by the term $-\alpha_2 \nabla p_1$. Finally, $\mathbf{f}_r$ is the force of additional influence on the inclusions because of gradients in the average velocity field of the supporting phase (Magnus or Zhukovskii force). For the forces indicated in (5.6) the following relationships can be written:

$$\mathbf{f}_f = \alpha_2 \rho_1^\circ K(\mathbf{v}_1 - \mathbf{v}_2), \qquad K = K(\alpha_2, \quad |\mathbf{v}_1 - \mathbf{v}_2|, \quad \mu_1, \mu_2, a, \ldots)$$

$$\mathbf{f}_m = \alpha_2 \rho_1^\circ \chi' \frac{d_2}{dt}(\mathbf{v}_1 - \mathbf{v}_2), \qquad \mathbf{f}_r = \alpha_2 \rho_1^\circ \chi'' (\mathbf{v}_1 - \mathbf{v}_2) \times \operatorname{rot} \mathbf{v}_1 \tag{5.9}$$

The influence of particle shape, their interaction and other refinements in the expressions of forces $\mathbf{f}_f$, $\mathbf{f}_m$, $\mathbf{f}_r$ can be taken into account in the coefficients K, χ' and χ''.

The intensity of phase interaction strongly depends on the characteristic dimension of inclusions a and their number n in the unit volume of mixture, so that

$$\alpha_2 = \eta n a^3 \tag{5.10}$$

Here η is a nondimensional coefficient which is determined by the shape of the particles ($\eta = 4\pi / 3$ for spherical particles of radius a). The quantity n can change due to convection and also as a result of the processes of fracture, agglomeration and formation of new particles. This is characterized by the quantity ψ in the equation for the number of particles

$$\frac{\partial n}{\partial t} + \nabla (n \mathbf{v}_2) = \psi \tag{5.11}$$

In the absence of fracture, agglomeration and formation of new particles ($\psi = 0$) and under the condition of incompressibility of the mass of inclusions ($\rho_2{}^\circ = \text{const}$), Eq. (5.11) is a consequence of the equation for the mass of the second phase for the condition $a = \text{const}$.

6. Energy transitions in phase transformations. In order to represent more concretely the exchange of energy between phases because of phase transformations, let us examine the elementary volume $dV = dV_1 + dV_2$ of the medium at rest (Fig. 1) in the vicinity of the interphase boundary Σ at the instants t and $t + dt$ when the boundary Σ is displaced to the position Σ as a result of the transition $2 \to 1$. In this process the mass of the second phase

$$J_{21} dV\, dt \quad (J_{21} dV\, dt \ll \rho_i{}^\circ dV_i = \rho_i dV)$$

which is located at the instant t between the surfaces Σ and Σ_2, moves (expands) to the position between the surfaces Σ' and Σ_1.

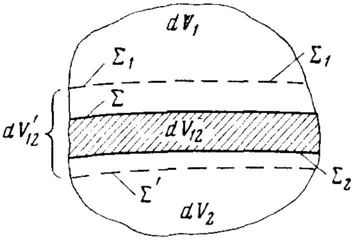

Fig. 1

For simplicity we will not examine the interaction between the selected volume dV and the external medium. It will also be assumed that the phases are homogeneous throughout to the very interphase boundary.

Taking into accunt (5.4), we write the equations of heat influx for three fixed masses:

1) Mass of material which at the instant t was in the form of the first phase (above the surface Σ).

2) Mass of material which at the instant $t + dt$ was in the form of the second phase (under the surface Σ').

3) Mass of material which has undergone the transition $2 \to 1$ (at the instant t it occupied the volume between Σ and Σ_2).

Then we have

$$\begin{gathered}
\rho_1 dV du_1 = q_{21} dt - J_{21} dV\, dt \quad x_{1,21} + J_{21}\, dV\, dt\, \varkappa_1 p \Delta\vartheta \\
(\rho_2\, dV - J_{21}\, dV\, dt)\, du_2 = q_{12}\, dt \ - J_{21} dV dt \quad x_{2,21} + J_{21} dV\, dt\, \varkappa_2 p \Delta\vartheta \\
J_{21}\, dV\, dt \ (u_1 + du_1 \ - u_2) = J_{21}\, dV\, dt\, (x_{1,21} + x_{2,21}) - J_{21} \ dV\, dt \ \ p\Delta\vartheta \\
(\varkappa_1 + \varkappa_2 = 1, \ \Delta\vartheta = 1 / \rho_1{}^\circ - 1 / \rho_2{}^\circ)
\end{gathered} \tag{6.1}$$

Here the last terms correspond to the work of internal forces during the change in volume. With accuracy to small terms of higher order we obtain from (6.1)

$$\begin{aligned} \rho_1 du_1 &= q_{21}dt - J_{21}x_{1,21}dt + J_{21}\varkappa_1 p\Delta\vartheta \\ \rho_2 du_2 &= -q_{21}dt - J_{21}x_{2,21}dt + J_{21}\varkappa_2 p\Delta\vartheta \\ u_1 - u_2 &= x_{1,21} + x_{2,21} - p\Delta\vartheta \end{aligned} \qquad (6.2)$$

Analogous relationships can be obtained if the transition $1 \rightarrow 2$ is examined. Introducing enthalpies of phases $i_i = u_i + p_i/\rho_i^\circ$, it follows from the last equation of (6.2)

$$x_{1,21} + x_{2,21} = i_1 - i_2, \qquad x_{1,12} + x_{2,12} = i_2 - i_1 \qquad (6.3)$$

Generalizing the obtained expressions to the case of the transition $1 \rightarrow 2$, the first two Eqs. (6.2) can be represented in the form

$$\begin{aligned} d(\rho_1 u_1)/dt &= q_{21} + J_{21}(u_1 - x_{1,21}) - J_{12}(u_1 + x_{1,12}) + \\ &\quad + (J_{21} - J_{12})\varkappa_1 p\Delta\vartheta \\ d(\rho_2 u_2)/dt &= -q_{21} - J_{21}(u_2 + x_{2,21}) + J_{12}(u_2 - x_{2,12}) + (J_{21} - J_{12})\varkappa_2 p\Delta\vartheta \\ &(\varkappa_1 + \varkappa_2 = 1, \ \Delta\vartheta = 1/\rho_1^\circ - 1/\rho_2^\circ) \end{aligned} \qquad (6.4)$$

In this manner in the transition $2 \rightarrow 1$ the internal energy of the second phase decreases by the quantity $J_{21}(u_2 + x_{2,21})$. The $J_{21}(u_1 - x_{1,21})$ part of this energy is transformed into the internal energy of the first phase. The remaining part $p\Delta\vartheta$ is distributed among the phases (in proportion to $\varkappa_i$) in the form of work of internal forces. As a result the quantities u_{12} and u_{21} can be written in the form

$$\begin{aligned} u_{12} &= u_2 - x_{2,12} + \varkappa_2 p\Delta\vartheta = u_1 + x_{1,12} - \varkappa_1 p\Delta\vartheta \\ u_{21} &= u_2 + x_{2,21} - \varkappa_2 p\Delta\vartheta = u_1 - x_{1,21} + \varkappa_1 p\Delta\vartheta \end{aligned} \qquad (6.5)$$

For further progress it is necessary to know which part of energy $i_1 - i_2$ is expended or absorbed separately by the first and second phases in the transformation $2 \rightarrow 1$ (or $1 \rightarrow 2$) of some mass of the second (first) phase, i.e. it is necessary to specify the relationships for u_{12} and u_{21} (see (3.6)) or $x_{1,12}$ and $x_{1,21}$. Following the development in [12], we take

$$x_{i,ij} = i_{js} - i_i \qquad (6.6)$$

here the index s refers to the state of saturation

$$i_{js} = i_j(p, T_s(p)) \qquad (6.7)$$

and T_s is the temperature for the equilibrium transition $1 \leftrightarrows 2$. Taking into account (6.3), it follows from (6.6) that

$$x_{2,21} = i_{1s} - i_2, \quad x_{1,21} = i_1 - i_{1s}, \quad x_{2,12} = i_2 - i_{2s}, \quad x_{1,12} = i_{2s} - i_1 \qquad (6.8)$$

Relationships (6.6) or (6.8) actually postulate the fact that in the transformation $i \rightarrow j$ the ith component directly expends or absorbs the energy which is necessary at the ambient pressure in order to bring the mass which is undergoing transition to the state which has the form of the jth phase under consitions of saturation. The remaining energy which is necessary to take this mass from the saturated state to the existing state of the jth phase is expended (absorbed) by the jth phase itself.

The quantity $q_{12} = -q_{21}$, which describes the intensity of the heat transfer between the components due to inequality in their temperatures, in many cases (when the radiant heat transfer is insignificant) can be assumed to be proportional to the temperature difference between the phases

$$q_{12} = \beta(T_1 - T_2) \qquad (6.9)$$

here β depends on the shape, the size of inclusions, the physical properties and the relative motion of the phases.

7. The complete system of equations of motion for the dispersed mixture. Further development of a concrete model of motion of a liquid or gas mixture with foreign inclusions (Sect. 5) is tied to explicit determinations of the capacity of internal forces for work in the unit mass of the ith phase A_i (see (3.7)). This capacity, generally speaking, is represented in the form (summation over superscripts)

$$A_i = \frac{\sigma_i^{\circ kl} e_i^{\circ kl}}{\rho_i^\circ} = \frac{p_i}{\rho_i^{\circ 2}} \frac{d_i \rho_i^\circ}{dt} + \frac{\tau_i^{\circ kl} e_i^{\circ kl}}{\rho_i^\circ} \tag{7.1}$$

The second term gives the dissipated energy in the ith phase due to internal viscous forces which arise because of gradients in the velocity field $\mathbf{v}_i$ and also because of interaction with the other phase. Since the direct determination of the actual tensor of deformation velocities is difficult in this case, it is appropriate to take advantage of some assumptions which result from the analysis of the motion of inclusions in the supporting flow of the medium and an analysis of (3.6). For example

$$\rho_i A_i = \frac{\alpha_i p_i}{\rho_i^\circ} \frac{d_i \rho_i^\circ}{dt} + \nu_i f_{ji} (\mathbf{v}_j - \mathbf{v}_i) + \tau_i^{kl} e_i^{kl} + J_{ji} \frac{(\mathbf{v}_{ji} - \mathbf{v}_i)^2}{2} - J_{ij} \frac{(\mathbf{v}_{ij} - \mathbf{v}_i)^2}{2}$$

$$\nu_1 + \nu_2 = 1 \tag{7.2}$$

Coefficients ν_i, first introduced on the basis of formal considerations in [19], show the fraction of kinetic energy of the mixture which can be dissipated as a result of interaction between components and which transfers directly into the internal energy of the ith phase.

We note in connection with this that components of the interphase force $\mathbf{r}_{12}$ (see (5.7)-(5.9)), the force connected with the effect of associated masses $\mathbf{f}_m$, the Magnus force $\mathbf{f}_r$, and also the Archimedean force lead directly to a transfer of a part of the kinetic energy of macroscopic motion not into internal (thermal) energy phases, but into the kinetic energy of small-scale (see note before (2.6)) flows within and near inclusions. As we pointed out earlier, the latter energy is not taken into account in existing theories of interpenetrating motion. In (7.2) $\mathbf{f}_m$ and $\mathbf{f}_r$ enter as dissipative forces.

In many cases, where it is possible to neglect dislocation deformations of inclusions (particles, drops, bubbles) with respect to the supporting phase, the following relationships are valid:

$$\nu_2 = 0, \quad \nu_1 = 1, \quad \mathbf{v}_{21} = \mathbf{v}_{12} = \mathbf{v}_2 \tag{7.3}$$

In this manner, taking into account (3.7), (5.1), (5.2), (5.5), (5.6), (6.8) and (7.2), the system of equations of two-phase motion (3.2) for dispersed heterogeneous mixtures as indicated in Sect. 5, takes the form

$$\frac{\partial \rho_1}{\partial t} + \nabla (\rho_1 \mathbf{v}_1) = J_{21} - J_{12}$$

$$\frac{\partial \rho_2}{\partial t} + \nabla (\rho_2 \mathbf{v}_2) = J_{12} - J_{21}, \qquad \frac{\partial n}{\partial t} + \nabla (n \mathbf{v}_2) = \psi$$

$$\rho_1 \frac{d_1 \mathbf{v}_1}{dt} = -\alpha_1 \nabla p + \nabla^k \boldsymbol{\tau}_1^k - \mathbf{f}_{12} + J_{21} (\mathbf{v}_{21} - \mathbf{v}_1) - J_{12} (\mathbf{v}_{12} - \mathbf{v}_1) + \rho_1 \mathbf{F}_1$$

$$\rho_2 \frac{d_2 \mathbf{v}_2}{dt} = -\alpha_2 \nabla p + \mathbf{f}_{12} - J_{21} (\mathbf{v}_{21} - \mathbf{v}_2) + J_{12} (\mathbf{v}_{12} - \mathbf{v}_2) + \rho_2 \mathbf{F}_2$$

$$\rho_1 \frac{d_1 u_1}{dt} = \frac{\alpha_1 p}{\rho_1^\circ} \frac{d_1 \rho_1^\circ}{dt} + \nu_1 \dot{\mathbf{f}}_{12} (\mathbf{v}_1 - \mathbf{v}_2) + \boldsymbol{\tau}_1^{kl} e_1^{kl} + J_{21} \frac{(\mathbf{v}_{21} - \mathbf{v}_1)^2}{2} -$$

$$- J_{12} \frac{(\mathbf{v}_{12} - \mathbf{v}_1)^2}{2} + J_{21} (i_{1s} - i_1) - J_{12} (i_{2s} - i_1) - q_{12} - \nabla \mathbf{q}_1 + \rho_1 Q_1$$

$$\rho_2 \frac{d_2 u_2}{dt} = \frac{\alpha_2 p}{\rho_2^\circ} \frac{d_2 \rho_2^\circ}{dt} + \nu_2 \mathbf{f}_{12} (\mathbf{v}_1 - \mathbf{v}_2) - J_{21} \frac{(\mathbf{v}_{21} - \mathbf{v}_2)^2}{2} + J_{12} \frac{(\mathbf{v}_{12} - \mathbf{v}_2)^2}{2} -$$

$$- J_{21} (i_{1s} - i_2) + J_{12} (i_{2s} - i_2) + q_{12} + \rho_2 Q_2$$

$$p_1 (\rho_1^\circ, T_1) = p_2 (\rho_2^\circ, T_2) = p \qquad (7.4)$$

$$(\alpha_1 + \alpha_2 = 1,\ \rho_1^\circ = \rho_1 / \alpha_1,\ \rho_2^\circ = \rho_2 / \alpha_2,\ \nu_1 + \nu_2 = 1,\ a^3 = \alpha_2 / \eta n)$$

Relationships (5.1)–(5.4) (or their generalizations) characterizing the physical-chemical properties of phases, relationships (or their generalizations) of the type (5.7)–(5.9), (6.9), and (7.3) describing the interaction of phases, together with kinetic equations for ψ, J_{12} and J_{21} with specified external influences ($\mathbf{F}_1$, $\mathbf{F}_2$, Q_1, Q_2) close the system of equations (7.4) in the region of continuous flow.

From (7.4) we can obtain equations of momentum and energy of each phase in the form (3.2), but in a more explicit representation

$$\rho_i \frac{d_i \mathbf{v}_i}{dt} = \nabla^k \boldsymbol{\sigma}_i^{*k} + \mathbf{r}_{ji}^* + J_{ji} (\mathbf{v}_{ji} - \mathbf{v}_i) - J_{ij} (\mathbf{v}_{ij} - \mathbf{v}_i) + \rho_i \mathbf{F}_i \qquad (7.5)$$

$$\rho_i \frac{d_i}{dt} \left(u_i + \frac{v_i^2}{2} \right) = \nabla^k \boldsymbol{\sigma}_i^{*k} \mathbf{v}_i - p \frac{\partial \alpha_i}{\partial t} + \mathbf{f}_{ji} \mathbf{v}_i + \nu \mathbf{f}_{ji} (\mathbf{v}_j - \mathbf{v}_i) +$$

$$+ J_{ji} \left(i_{is} - e_i + \frac{v_{ji}^2 - v_i^2}{2} \right) - J_{ij} (i_{js} - e_i + \frac{v_{ij}^2 - v_i^2}{2}) + q_{ji} - \nabla \mathbf{q}_i +$$

$$+ \rho_i (\mathbf{F}_i \mathbf{v}_i + Q_i)$$

$$\sigma_1^{*kl} = - \alpha_1 p \delta^{kl} + \tau_1^{kl}, \quad \sigma_2^{*kl} = - \alpha_2 p \delta^{kl}, \quad \mathbf{r}_{12}^* = - \mathbf{r}_{21}^* = \mathbf{f}_{12} + p \nabla \alpha_2$$

$$(i = 1,2; \quad i \neq j)$$

This system corresponds to interpenetrating motion of two interacting continuous media in which the reduced tensors of surface forces σ_1^{*kl} and σ_2^{*kl}, the work of these forces, the reduced forces of interaction r_{12}^* and other terms which describe the exchange of mass, momentum, and energy have been formally determined.

Taking into account (7.5) we can write in a more explicit form the integral equations for the mixture (2.5) and (2.7) which are needed in the derivation of relationships on the surface of the discontinuity

$$\int_V \frac{\partial}{\partial t} (\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2)\, dV = - \int_S (\rho_1 \mathbf{v}_1 v_1^n + \rho_2 \mathbf{v}_2 v_2^n)\, dS +$$

$$+ \int_S (\boldsymbol{\sigma}_1^{*n} + \boldsymbol{\sigma}_2^{*n})\, dS + \int_V (\rho_1 \mathbf{F}_1 + \rho_2 \mathbf{F}_2)\, dV$$

$$\int_V \frac{\partial}{\partial t} \left[\rho_1 \left(u_1 + \frac{v_1^2}{2} \right) + \rho_2 \left(u_2 + \frac{v_2^2}{2} \right) \right] dV = - \int_S \left[\rho_1 \left(u_1 + \frac{v_1^2}{2} \right) v_1^n + \right.$$

$$\left. + \rho_2 \left(u_2 + \frac{v_2^2}{2} \right) v_2^n \right] dS + \int_S (\boldsymbol{\sigma}_1^{*n} \mathbf{v}_1 + \boldsymbol{\sigma}_2^{*n} \mathbf{v}_2)\, dS - \int_S (q_1^n + q_2^n)\, dS +$$

$$+ \int_V [\rho_1 (\mathbf{F}_1 \mathbf{v}_1 + Q_1) + \rho_2 (\mathbf{F}_2 \mathbf{v}_2 + Q_2)]\, dV$$

$$(\sigma_1^{*kl} = - \alpha_1 p \delta^{kl} + \tau_1^{kl}, \quad \sigma_2^{*kl} = - \alpha_2 p \delta^{kl}) \qquad (7.6)$$

8. Conditions on the surfaces of discontinuity. In the flow of a heterogeneous mixture, regions can arise (shock waves, wall layer, contact surfaces) in which the parameters of the medium change substantially over distances of the order of dimensions of the inclusions or smaller (from the point of view of the continuous medium these distances are equal to zero). In these regions the model of a continuous heterogeneous medium and the differential equations (3.2) or (7.4), which are based on this model, are meaningless. Therefore, as it is usually done, it is necessary to introduce into the analysis surfaces of discontinuities in flow parameters. On both sides of these surfaces the equations of continuous motion are fulfilled. The basic conditions on the surface of discontinuity Σ are obtained, in a system of coordinates connected with this surface, proceeding from integral equations of Sect. 2, which are applied to a small cylindrical volume at rest with respect to Σ and with bases parallel to Σ located on opposite sides of it. In this connection it will be assumed that processes of heat and mass transfer on these surfaces do not have time to occur. We omit the usual calculations for such cases [15], and obtain from (2.1) and (7.6)

$$\begin{gathered}(\rho_1 v_1^n)^+ = (\rho_1 v_1^n)^- = j_1, \qquad (\rho_2 v_2^n)^+ = (\rho_2 v_2^n) = j_2 \\ j_1 [\mathbf{v}_1] + j_2 [\mathbf{v}_2] + [\overset{*}{\boldsymbol{\sigma}}{}_1^n] + [\overset{*}{\boldsymbol{\sigma}}{}_2^n]^- = 0 \qquad (8.1) \\ j_1 [u_1 + {}^1\!/_2 v_1^2] + j_2 [u_2 + {}^1\!/_2 v_2^2] + [\overset{*}{\boldsymbol{\sigma}}{}_1^n \mathbf{v}_1] + [\overset{*}{\boldsymbol{\sigma}}{}_2^n \mathbf{v}_2] + [q_1^n] + [q_2^n] = 0 \\ ([\ \ ] = (\ )^+ - (\ )^-)\end{gathered}$$

Here the superscripts plus and minus refer to conditions on opposite sides of the surface of discontinuity. The symbols n and τ are the normal and tangential directions to the surface Σ. In order to close system (8.1), i.e. to make it possible to determine all parameters of flow on one side of Σ from parameters on the other side, it is necessary to use data on physical-mechanical properties of phases and data on their mutual interactions in the narrow regions of interest.

For heterogeneous systems indicated in Sect. 5, taking into account expressions (7.6) for reduced tensors $\overset{*}{\sigma}{}_i^{kl}$, the expressions (8.1) assume the following form if we neglect effects of viscosity and thermal conductivity outside of the surface of discontinuity:

$$\begin{gathered}j_1 [v_1^n] + j_2 [v_2^n] + [p] = 0, \qquad j_1 [v_1^\tau] + j_2 [v_2^\tau] = 0 \\ j_1 [i_1 + {}^1\!/_2 v_1^2] + j_2 [i_2 + {}^1\!/_2 v_2^2] = 0 \qquad (8.2)\end{gathered}$$

One of possible additional relationships was obtained in [20] through the examination of the limit for a sequence of continuous solutions which tend to approach the discontinuous one. As a result the following expression is valid instead of the third equation of (8.2)

$$j_1 [i_1 + {}^1\!/_2 v_1^2] = 0, \qquad j_2 [i_2 + {}^1\!/_2 v_2^2] = 0 \qquad (8.3)$$

Together with equations of state of phases (5.4) and Eq. (5.6) (equilibrium of phases with respect to pressure) it is necessary to utilize relationships which characterize the exchange of momentum between components in the jump, in order to close the system of equations (8.2) and (8.3). In [11] for the case of gas-incompressible particles mixture, the following relations are used:

$${}^1\!/_2 [v_2^2] + [p] / \rho_2^\circ = 0, \qquad j_1 [v_1^\tau] = j_2 [v_2^\tau] = 0 \qquad (8.4)$$

Furthermore, a classification of discontinuities was made.

9. Thermodynamic analysis. Generation of entropy. Proceeding from the assumption of additivity of the internal energy of the mixture with respect to masses of components (Sect. 2) and the assumption of local equilibrium within the boundaries of a phase, we can introduce a function of specific internal energy for the mixture

$$\rho u (\rho_1^\circ, \rho_2^\circ, T_1, T_2, \alpha) = \rho_1 u_1 (\rho_1^\circ, T_1) + \rho_2 u_2 (\rho_2^\circ, T_2) \qquad (9.1)$$

and an analogous specific entropy of the mixture

$$\rho s(\rho_1{}^\circ, \rho_2{}^\circ, T_1, T_2, \alpha) = \rho_1 s_1(\rho_1{}^\circ, T_1) + \rho_2 s_2(\rho_2{}^\circ, T_2) \tag{9.2}$$

In this connection the introduction of the concept of temperature T for the mixture in the case where $T_1 \neq T_2$, as it was done in [7, 8], is incorrect because in spite of local equilibrium of each phase the mixture will not be locally in equilibrium. Therefore, expressions of the type $u(\rho, T)$ and $s(\rho, T)$ for heterogeneous mixtures where $T_1 \neq T_2$ are devoid of meaning. For example, it is immediately evident that this leads either to the neglect of some degrees of freedom of the behavior of the mixture, or it requires [7] for T to be regarded as a functional which is determined by the history of the process.

We obtain the dissipation function for the medium described by Eqs. (7.4) or (7.5), i.e. the function which gives the generation of entropy of the mixture due to internal processes for a fixed mass of the medium. We follow [21] and determine the substantive derivative for any function (for example s) which is additive with respect to masses of components. For the volume V which was examined in Sect. 2 we have

$$\int\limits_V \frac{\partial}{\partial t}(\rho s)\, dV = -\int\limits_S \sum_{j=1}^{m} \rho_j s_j v_j{}^n\, dS + \int\limits_V \rho \frac{ds}{dt}\, dV \tag{9.3}$$

Here the last term gives the change in s which is not related to mass transfer. Changing over to the differential form and utilizing (9.2) and the equations of conservation of mass of each phase (3.2), we obtain for the case $m = 2$

$$\rho \frac{ds}{dt} = \rho_1 \frac{d_1 s_1}{dt} + \rho_2 \frac{d_2 s_2}{dt} + (J_{12} - J_{21})(s_2 - s_1) \tag{9.4}$$

Actually (9.3) or (9.4) is the determination of the operator d/dt for any function which is additive with respect to masses entering into the mixture of phases.

The condition of local equilibrium within the boundaries of each phase leads to the Gibbs relationship (5.5) for the mass of each phase. Taking into account (1.2) and (5.6) this relationship can be presented in the form

$$\rho_i T_i \frac{d_i s_i}{dt} = \rho_i \frac{d_i u_i}{dt} - \frac{\alpha_i p_i}{\rho_i{}^\circ} \frac{d_i \rho_i{}^\circ}{dt} \tag{9.5}$$

We note that the requirement of local equilibrium of phases imposes in addition to the usual limitations an upper limit on the temperature difference of phases $|T_1 - T_2|$.

In this manner we have from (9.4), (9.5) and (7.4)

$$\begin{aligned}\rho \frac{ds}{dt} = {} & \frac{\rho_1 Q_1}{T_1} + \frac{\rho_2 Q_2}{T_2} - \frac{\nabla \mathbf{q}_1}{T_1} + \frac{\tau_1{}^{kl} e_1{}^{kl}}{T_1} + \mathbf{f}_{12}(\mathbf{v}_1 - \mathbf{v}_2)\left(\frac{\mathbf{v}_1}{T_1} + \frac{\mathbf{v}_2}{T_2}\right) + \\ & + q_{12}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) + J_{21}\left(s_1 - s_2 + \frac{i_{1s} - i_1}{T_1} - \frac{i_{2s} - i_2}{T_2} + \frac{(\mathbf{v}_{21} - \mathbf{v}_1)^2}{2T_1} - \right. \\ & \left. - \frac{(\mathbf{v}_{21} - \mathbf{v}_2)^2}{2T_2}\right) + J_{12}\left(s_2 - s_1 - \frac{i_{2s} - i_1}{T_1} + \frac{i_{2s} - i_2}{T_2} - \frac{(\mathbf{v}_{12} - \mathbf{v}_1)^2}{2T_1} + \frac{(\mathbf{v}_{12} - \mathbf{v}_2)^2}{2T_2}\right)\end{aligned} \tag{9.6}$$

The quantity ds/dt can be represented in the form of a sum of two terms

$$\frac{ds}{dt} = \frac{d^{(e)} s}{dt} + \frac{d^{(i)} s}{dt} \tag{9.7}$$

The first term determines the entropy increase of the mixture as a result of influx of entropy from outside because of energy exchange with the external medium. The second term (always positive) represents the entropy increase due to irreversible internal processes within the phase or between phases

$$\rho \frac{d^{(e)} s}{dt} = \frac{\rho_1 Q_1}{T_1} + \frac{\rho_2 Q_2}{T_2} - \nabla \left(\frac{\mathbf{q}_1}{T_1} \right) \tag{9.8}$$

$$\rho \frac{d^{(i)} s}{dt} = \sigma = \mathbf{J}_T \mathbf{X}_T + J_\tau{}^{kl} X_\tau{}^{kl} + \mathbf{J}_f \mathbf{X}_f + J_q X_q + J_{21} X_{21} + J_{12} X_{12}$$

The generation of entropy σ is as usual represented by the sum of products of thermodynamic forces X

$$\mathbf{X}_T = \nabla \left(\frac{1}{T_1} \right), \ \mathrm{X}_\tau^{kl} = \frac{e_1^{kl}}{T_1}, \ \mathbf{X}_f = (\mathbf{v}_1 - \mathbf{v}_2) \left(\frac{\mathbf{v}_1}{T_1} + \frac{\mathbf{v}_2}{T_2} \right), \ X_q = \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$$

$$X_{21} = s_1 - s_2 + \frac{i_{1s} - i_1}{T_1} - \frac{i_{1s} - i_2}{T_2} + \frac{(\mathbf{v}_{21} - \mathbf{v}_1)^2}{2T_1} - \frac{(\mathbf{v}_{21} - \mathbf{v}_2)^2}{2T_2} \tag{9,9}$$

$$X_{12} = s_2 - s_1 - \frac{i_{2s} - i_1}{T_1} + \frac{i_{2s} - i_2}{T_2} - \frac{(\mathbf{v}_{12} - \mathbf{v}_1)^2}{2T_1} + \frac{(\mathbf{v}_{12} - \mathbf{v}_2)^2}{2T_2}$$

and thermodynamic fluxes

$$\mathbf{J}_T = \mathbf{q}_1, \qquad J_\tau{}^{kl} = \tau_1{}^{kl}, \qquad \mathbf{J}_f = \mathbf{f}_{12}, \quad J_q = q_{12}, \quad J_{21}, J_{12} \tag{9.10}$$

In the presence of finite coupling between thermodynamic forces X and thermodynamic fluxes J we can consider the quantity σ as a dissipation function of X or J. On the basis of (9.8)–(9.10) we can for example propose linear relationships between them, special cases of which are (5.2), (5.3), (6.9) and in some cases the first equation of (5.9).

We introduce the notation

$$\theta_1 = (T_1 - T_s) / T_s, \qquad \theta_2 = (T_2 - T_s) / T_s \tag{9.11}$$

In case of small deviations of phase temperatures T_1 and T_2 from the saturation temperature $T_s(p)$, i.e. when $|\theta_1| \ll 1$, $|\theta_2| \ll 1$ the expressions for thermodynamic forces X_{12} and X_{21} connected with phase transformations simplify. In this case

$$i_1 = i_{2s} + l + c_{p1} T_s \theta_1, \qquad i_2 = i_{2s} + c_{p2} T_s \theta_2$$

$$s_1 = s_2 + l / T_s + c_{p1} \theta_1, \qquad s_2 = s_{2s} + c_{p2} \theta_2 \tag{9.12}$$

here $l(p)$ is the heat of transition 2 → 1, c_{p1} and c_{p2} are heat capacities at constant pressure of the first and second phases. Taking into account (9.12), it follows from (9.9) that:

$$X_{21} = \frac{l\theta_2}{T_s} + \frac{(\mathbf{v}_{21} - \mathbf{v}_1)^2}{2T_s} - \frac{(\mathbf{v}_{21} - \mathbf{v}_2)^2}{2T_s}, \ X_{12} = -\frac{l\theta_1}{T_s} + \frac{(\mathbf{v}_{12} - \mathbf{v}_1)^2}{2T_s} + \frac{(\mathbf{v}_{12} - \mathbf{v}_2)^2}{2T_s} \tag{9.13}$$

For the velocities of phase transitions we can present the following linear kinetic equations:

$$J_{21} = L_{21} X_{21} \quad (X_{21} > 0), \qquad J_{21} = 0 \quad (X_{21} \leqslant 0)$$

$$J_{12} = L_{12} X_{12} \quad (X_{12} > 0), \qquad J_{12} = 0 \quad (X_{12} \leqslant 0) \tag{9.14}$$

$$(L_{21} \geqslant 0, \quad L_{12} \geqslant 0)$$

The following inequalities are always valid:

$$J_{21} \geqslant 0, \qquad J_{12} \geqslant 0, \qquad J_{21} X_{21} \geqslant 0, \qquad J_{12} X_{12} \geqslant 0 \tag{9.15}$$

We note that the kinetic equations which were utilized usually have the form [15, 22]

$$J_{ij} = J_{ij} (l\theta_i / T_s) \tag{9.16}$$

i.e. they do not take into account dissipation effects due to velocity differences between the phases. It follows from (9.13) in the case (7.3) that with other conditions being equal these effects accelerate the transfer to the high temperature medium and retard it to the low temperature one.

In an analogous manner many available equations can be generalized to the case of a mixture with more than two components ($m > 2$) (a particular case of such a generalization is given in [23]). The possible multiparameter character of phases (chemical reactions within the phase, electromagnetic effects, etc.) can also be taken into account.

In conclusion we mention the monograph [24] just published which gives a detailed survey of literature and a derivation for the fundamental equations of the mechanics of mixtures for both the elastic and liquid phases without phase transitions.

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